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ON THE STRONG STABILITY OF RESONANT SYSTEMS UNDER PARAMETRIC PERTURBATIONS

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An autonomous system of differential equations with holomorphic right-hand sides that continuously depend on a parameter is considered. It is assumed that in the considered region of parameter values the first approximation system has n pairs of pure imaginary roots.

The stability properties of a system may undergo various changes when parameter μ is varied near its resonance value μ_0 at which the system has an internal resonance. The problem of strong stability is posed in the case when the character of the system stability at point μ_0 remains unchanged in some neighborhood of that point.

A normal form, continuous with respect to μ of the system is derived. The form, unlike the usual and normal form [1, 2], does not change its structure at transition of the system through resonance (*). Conditions of strong asymptotic stability and instability are obtained for resonance of odd order. Cases of "explosive instability" in which the asymptotic stability in the resonance point neighborhood changes into instability at point μ_0 are separated out.

The concept of the present work is akin to that of [3], where the problem of stability in the presence of parametric perturbations is stated and substantiated. Several aspects of this problem and its generalization were discussed in [4]. The allied question of the form of the problem of the difference between dangerous and safe limits of the stability region was considered in [5].

1. Statement of the problem. Let us consider the system of differential equations

$$z^{\cdot} = P(\mu) z + \sum_{l=k\geqslant 2}^{\infty} Z^{(l)}(z, \mu), \quad \mu \in (\mu_1, \mu_2) = D \tag{1.1}$$

where z is an r-dimensional vector, $P(\mu)$ is an $(r \times r)$-matrix, $Z^{(l)}(z, \mu)$ are vector-forms of l-th order with respect to z. Matrix $P(\mu)$ and the coefficients of form $Z^{(l)}(z, \mu)$ are continuous functions of parameter μ.

We shall call system (1.1) stable (unstable) at point $\mu_0 \in D$, if the zero solution of that system for $\mu = \mu_0$ is Liapunov stable (unstable).

The considered problem is associated with the analysis of the effect of small variation of the parameter on the stability properties of system (1.1).

Definition 1.1. System (1.1) which is stable (unstable) at point $\mu_0 \in D$ is called strongly stable (unstable) at that point, if there exists such ε-neighborhood

*) Gol'tser, Ia. M. On the transformation of a particular system of differential equations in the presence of resonance. Transact. of the Seminar on the Theory of Motion Stability, № 2, Alma-Ata, 1969.

$U_\varepsilon(\mu_0) \subset D$ of point μ_0 that system (1.1) is stable (unstable) at every point $\mu \in U_\varepsilon(\mu_0)$.

Note 1.1. The term "strong stability" conforms to the terminology used in investigations of linear periodic Hamiltonian systems for small (but more general than parametric) perturbations of the Hamiltonian [6].

System (1.1) is considered below on the assumption that matrix $P(\mu)$ has in region D n pairs of continuous with respect to μ pure imaginary eigenvalues, and is reducible in D to a diagonal matrix which is continuous with respect to μ (The question of the normal form of matrices that depend on a parameter was investigated in [7]).

We set $r = 2n$ and denote the eigenvalues of matrix $P(\mu)$ by $\pm\lambda_s(\mu)$, $s = 1, 2, \ldots, n$, $\lambda_s^2(\mu) < 0$ $(\forall\mu \in D)$, and by R_n^+ the set of n-dimensional vectors whose components are nonnegative integers. If $m \in R_n^+$, then $m = (m_1, \ldots, m_n)$, $m_s \in R_1^+$, and $|m| = m_1 + \ldots + m_n$.

Definition 1.2. At point $\mu_0 \in D$ system (1.1) has a j-th order internal resonance, if there exists vector $m \in R_n^+$ with m_s relatively prime, such that when $\mu = \mu_0$

$$\langle m, \lambda^\circ\rangle = \sum_{s=1}^{n} m_s\lambda_s^\circ = 0, \quad |m| = j, \quad \lambda^\circ = \lambda(\mu_0) = (\lambda_1^\circ, \ldots, \lambda_n^\circ) \qquad (1.2)$$

An effective method of investigating the stability of system (1.1) with fixed values of the parameter (either resonant or nonresonant) consists of its preliminary reduction to the normal form up to terms of a reasonably high order [1, 2]. If point μ_0 is nonresonant, such normalization in a fairly small neighborhood of it is continuous with resect to μ, and one can expect that the system stability properties are retained in that neighborhood. When parameter μ passes through its resonance value μ_0, the structure of the usual normal form considerably changes. Hence for the considered class of systems the problem of strong stability in the neighborhood of resonant values of the parameter is of considerable interest. The question of existence for system (1.1) of a normal form continuous with respect to μ, which arises in this connection, is considered below.

2. The continuous normal form. On the assumptions made in Sect. 1, system (1.1) may be written as (the upper dash denotes a complex-conjugate quantity)

$$x^\cdot = \Lambda(\mu)x + \sum_{l=k\geqslant 2}^{\infty} X^{(l)}(x, y, \mu) \qquad (2.1)$$

$$y^\cdot = -\Lambda(\mu)y + \sum_{l=k\geqslant 2}^{\infty} Y^{(l)}(x, y, \mu)$$

where $x = \bar{y} = (x_1, \ldots, x_n)$, E is a unit $(n \times n)$-matrix, and $\Lambda(\mu) = \lambda(\mu)E$. We represent the components of n-dimensional vector-forms $X^{(l)} = Y^{(l)}$ in the form

$$X_s^{(l)}(x, y, \mu) = \sum_{|p|+|q|=l} a_{p,q}^s(\mu)x^p y^q, \quad p, q \in R_n^+, \; x^p = x_1^{p_1}\ldots x_n^{p_n}$$

Functions $\lambda_s(\mu)$ and $a_{p,q}^s(\mu)$ are continuous in D.

Let us consider system

$$u^\cdot = \Lambda(\mu)u + \sum_{l=k\geqslant 2}^{\infty} U^{(l)}(u, v, \mu) \qquad (2.2)$$

$$v^{\cdot} = -\Lambda(\mu)v + \sum_{l=k\geqslant 2}^{\infty} V^{(l)}(u, v, \mu)$$

$$U_s^{(l)} = \overline{V}_s^{(l)} = \sum_{|p|+|q|=l} \alpha_{p,q}^s(\mu) u^p v^q$$

and transformation

$$x = u + \sum_{l=k\geqslant 2}^{\infty} \Phi^{(l)}(u, v, \mu), \quad y = v + \sum_{l=k\geqslant 2}^{\infty} \overline{\Phi}^{(l)}(u, v, \mu) \quad (2.3)$$

$$\Phi_s^{(l)} = \sum_{|p|+|q|=l} A_{p,q}^s(\mu) u^p v^q$$

together with (2.1), and seek the simplest form of system (2.2) to which system (2.1) can be reduced by the transformation, continuous with respect to μ, in the form of formal series (2.3).

For the successive determination of forms $\Phi_s^{(l)}$ from (2.1) — (2.3) we obtain

$$\sum_{r=1}^{n} \lambda_r \left(\frac{\partial \Phi_s^{(l)}}{\partial u_r} u_r - \frac{\partial \Phi_s^{(l)}}{\partial v_r} v_r \right) = \lambda_s \Phi_s^{(l)} - U_s^{(l)} - F_s^{(l)} + X_{s*}^{(l)} \quad (2.4)$$

$$F_s^{(l)} = \sum_{r=1}^{n} \sum_{j=k}^{l-1} \frac{\partial \Phi_s^{(j)}}{\partial u_r} U_r^{(l-j+1)} + \frac{\partial \Phi_s^{(j)}}{\partial v_r} V_r^{(l-j+1)}$$

where $X_{s*}^{(l)}$ is an l-th order form in the expansion in series of functions

$$\sum_{j=k\geqslant 2}^{l} X_s^{(j)}(u + \ldots, v + \ldots)$$

For the simultaneous determination of coefficients of forms $\Phi_s^{(l)}$ and $U_s^{(l)}$ we obtain

$$\langle p - q - \delta_s, \lambda(\mu)\rangle A_{p,q}^s = a_{p,q}^{s*}(\mu) - f_{p,q}^s(\mu) - \alpha_{p,q}^s(\mu) \quad (2.5)$$

where $a_{p,q}^{s*}(\mu)$ and $f_{p,q}^s(\mu)$ ($|p| + |q| = l$) are coefficients of forms $X_{s*}^{(l)}$ and $F_s^{(l)}$, and $\delta_s = (0, \ldots, 1, \ldots, 0)$ is the basis vector in R_n^+.

Definition 2.1. The bivectors (p, q), $p \neq q + \delta_s$ and the related coefficients s-x in the equations of systems (2.1) and (2.2), and in the transformation (2.3) are called resonant, if there exists a $\mu_0 \in D$ such that

$$\langle p - q - \delta_s, \lambda(\mu_0)\rangle = 0$$

We denote the set of all resonant bivectors of the s-th equation by L_D^s. When $p = q + \delta_s$ for the bivectors (p, q) we have

$$\langle p - q - \delta_s, \lambda(\mu)\rangle \equiv 0 \quad (\forall \mu \in D)$$

which constitute the set L_0^s that determines the terms of the identical resonance. We denote $L^s = L_D^s \cup L_0^s$.

Let functions $a_{p,q}^s$ and $f_{p,q}^s$ in (2.5) be continuous in D. It is evident that when $(p, q) \in L^s$, then (2.5) has a solution that is continuous with respect to μ for any selected continuous functions $\alpha_{p,q}^s(\mu)$.

If $(p, q) \in L_D^s$, there exists a point $\mu_0 \in D$ such that $\langle p - q - \delta_s, \lambda\rangle \to 0$ when $\mu \to \mu_0$. For function $A_{p,q}^s(\mu)$ to be continuous in D it is necessary to determine function $\alpha_{p,q}^s(\mu)$ so that it is continuous in D, and that at each resonant point of region D the right-hand side of (2.5) is infinitely small of an order not less than $\langle p - q - \delta_s, \lambda(\mu)\rangle$. It is obvious that with such choice the structure of functions $\alpha^s_{p,q}$ and $A^s_{p,q}$ in the neighborhood of the resonance point is of the form

$$\begin{aligned} \alpha^s_{p,q}(\mu) &= a^{s*}_{p,q}(\mu_0) + f^s_{p,q}(\mu_0) - \varepsilon^s_{p,q}(\mu) \\ A^s_{p,q}(\mu) &= \langle p - q - \delta_s, \lambda(\mu)\rangle^{-1}\varepsilon^s_{p,q}(\mu), \quad (p, q) \in L_D{}^s \end{aligned} \tag{2.6}$$

where $\varepsilon^s_{p,q}$ is an arbitrary continuous function in D such that there exists the limit

$$\lim_{\mu\to\mu_0} A^s_{p,q}(\mu) = A^s_{p,q}(\mu_0) \tag{2.7}$$

The constant $A^s_{p,q}(\mu_0)$ can be arbitrarily chosen, and this can be used for further simplification of the system in the case of third order resonances.

If $(p, q) \in L_0{}^s$, then it is sufficient to set in (2.5)

$$\alpha^s_{q+\delta_s, q} = a^{s*}_{q+\delta_s, q}(\mu) + f^s_{p,q}(\mu) \tag{2.8}$$

Taking now into consideration that forms $X_{s*}{}^{(l)}$ and $F_s{}^{(l)}$ depend only on forms $\Phi_s{}^{(j)}$ and $U_s{}^{(j)}$, where $j < l$, and that $X_{s*}{}^{(l)} = X_s{}^{(l)}$ and $F_s{}^{(l)} = 0$ when $l = k$, it is possible to assert that functions $f^s_{p,q}$ and $a^s_{p,q}$ are continuous in D for all $l = k, k + 1, \ldots$ Hence the following theorem is valid.

Theorem 2.1. With the arbitrary selection in system (2.2) that are continuous with respect to μ it is possible to select for that system continuous resonant coefficients so that the continuous in D transformation (2.3), which transforms system (2.1) into (2.2), is obtained.

The continuous normal form is obtained by equating in (2.2) all nonresonant coefficients to zero. The structure of the continuous normal form is then

$$\begin{aligned} u_s{}^{\bullet} &= \lambda_s(\mu) + u_s \sum_{q=[k/2]\geqslant 1}^{\infty} \alpha^s_{q+\delta_s, q}\omega^q + \sum_{(p,q)\in L^s_D} \alpha^s_{p,q} u^p v^q \\ v_s{}^{\bullet} &= \bar{u}_s{}^{\bullet}, \quad \omega = (\omega_1, \ldots, \omega_n), \quad \omega_s = u_s v_s \end{aligned} \tag{2.9}$$

The following properties of the transformation and of the normal form, which will be used subsequently, should be noted. If $k \geqslant 2$ is the lowest order of nonlinear terms in (2.1), forms $\Phi_s{}^{(j)}$ and $U_s{}^{(j)}$ for $k \leqslant j \leqslant 2k - 2$ are determined independently of forms $\Phi_s{}^{(l)}$ and $U_s{}^{(l)}$ for $l < j$. This property holds in conventional normalization and is due to the structure of functions $X_{s*}{}^{(l)}$ and $F_s{}^{(l)}$ in (2.4).

Let us compare the structure of system (2.9) for some fixed μ_0 with that of the usual normal form at that point. We use the notation $L_{\mu_0}{}^s = \{(p, q) \mid \langle p - q - \delta_s, \lambda^\circ\rangle = 0\}$. It is clear that $L_{\mu_0}{}^s \subset L_D{}^s$.

In the case of conventional normalization of system (2.1) the s-th equation contains only terms corresponding to bivectors $(p, q) \in L_{\mu_0}{}^s \cup L_0{}^s$ when $\mu = \mu_0$, while system (2.9) contains in addition to these terms also those for which $(p, q) \in L_D{}^s \setminus L_{\mu_0}{}^s$.

The two forms coincide at point μ_0 only when $L_D{}^s \setminus L_{\mu_0}{}^s = \varnothing$, but in the neighborhood of point μ_0, where $\mu \neq \mu_0$, they are different. Conventional normalization is continuous only when $L_D{}^s = \varnothing$ (as noted in Sect. 1).

When solving the problem of strong stability at point μ_0 we consider a reasonably small ε-neighborhood of point μ_0 as the region $D = U_\varepsilon(\mu_0)$. We denote the deleted neighborhood $U_\varepsilon(\mu_0)$ of point μ_0 by $D^* = U_\varepsilon{}^*$.

Let point μ_0 be resonant whose unique lowest resonance is of order g. We assume (and this is the general case) that μ_0 is the isolated root of the equation

$$\langle m,\ \lambda(\mu)\rangle = 0 \quad \text{for } |m| = g$$

The number ε is assumed to be so small, that when this equation has in D other integral solutions, their norm is $|m| \gg g$.

Under these conditions the lowest resonating terms in D are of order $g-1$, hence the structures of the conventional and the continuous normal forms coincide (for $k < g-1$) to within terms of order $g-2$. If the solution of the problem of stability in region D is independent of terms of order higher than $g-2$, the presence of inner resonance at point μ_0 does not affect the solution of the problem of strong stability. To detect possible bifurcations of the properties of stability by the presence in the system of a resonance of order g, we assume that in (2.1) $k = g-1$, and that $k = 2N$.

On these assumptions, and restricting the reduction of the continuous normal form to terms of order up to and including the ($k+1$)-st, we obtain the following system:

$$u_s^{\cdot} = \lambda_s(\mu)u_s + \alpha_s(\mu)v^{m-\delta_s} + u_s\sum_{|p|=N}\alpha_s^p(\mu)\omega^p + O_\mu(\|u+v\|^{2N+2}) \qquad (2.10)$$

$$v_s^{\cdot} = \bar{u}_s^{\cdot}, \quad m,\ p\in R_n^+, \quad |m| = k+1 = 2N+1 \geqslant 3$$

where $\lambda_s(\mu)$, $\alpha_s(\mu)$, $\alpha_s^p(\mu)$ and O_μ are continuous in D.

For $\mu = \mu_0$ system (2.10) is the same as that analyzed in [2, 8] and is obtained from (2.9) by considering the structure of lower resonating terms similar to those in [2].

3. Some properties of the model system. Let us consider the model system in the k-th approximation

$$u_s^{\cdot} = \lambda_s(\mu)\,u_s + \alpha_s(\mu)\,v^{m-\delta_s}, \quad v_s^{\cdot} = -\lambda_s(\mu)v_s + \bar{\alpha}_s(\mu)\,u^{m-\delta_s} \qquad (3.1)$$

We introduce ancilliary angles $\varphi_s(\mu)$ setting

$$\sin\varphi_s = -a_s\,|\alpha_s|^{-1}, \quad \cos\varphi_s = b_s\,|\alpha_s|^{-1}, \quad \alpha_s = a_s + ib_s \qquad (3.2)$$

and identify these with points of the unit trigonometric circle. Obviously $\varphi_s(\mu)$ are continuous functions of μ in D.

Let d_s be the diameter of the circle drawn through point φ_s. Two incompatible dispositions of points φ_s are possible.

A. For any diameter d_s there exist points φ_j that lie on different sides of d_s.

B. There exists diameter d_s such that all φ_j points lie on one side of it.

Case A necessitates that $n \geqslant 3$, while case B always obtains when $n = 2$. If case A obtains at point μ_0 (of region D), we say that conditions $A(\mu)$ ($A(D)$) are satisfied. Similarly $B(\mu)$, $B(D)$.

The algebraic characteristic of case A is defined by the following lemma.

Lemma 3.1. For condition $A(\mu)$ to be satisfied it is necessary and sufficient if there exist such coefficients $\alpha_{s_1}(\mu)$, $\alpha_{s_2}(\mu)$ and $\alpha_{s_3}(\mu)$ that

$$\operatorname{sign} D_{s_1s_2} = \operatorname{sign} D_{s_2s_3} = -\operatorname{sign} D_{s_1s_3}, \quad D_{s_js_k} = \begin{vmatrix} a_{s_j} & a_{s_k} \\ b_{s_j} & b_{s_k}\end{vmatrix} \qquad (3.3)$$

The following lemma is also valid.

Lemma 3.2. For condition $B(\mu)$ to be satisfied it necessary and sufficient that there exists such numbering of angles $\varphi_s(\mu)$ for which

$$\varphi_1(\mu) \leqslant \varphi_2(\mu) \leqslant \ldots \leqslant \varphi_n(\mu) \leqslant \varphi_1(\mu) + \pi \tag{3.4}$$

With the use of inequalities (3.4) we subdivide case B into the following subcases:

B_1 $(\exists l \mid 1 \leqslant l \leqslant n-1)$ $(\varphi_1 = \ldots = \varphi_l < \varphi_{l+1} \leqslant \ldots \leqslant \varphi_n < \varphi_1 + \pi)$

B_2 $(\forall s)$ $(\varphi_s = \varphi_1)$

B_3 $(\exists k \mid 1 \leqslant k \leqslant n-1)$ $(\varphi_1 = \ldots = \varphi_k < \varphi_{k+1} = \ldots = \varphi_n = \varphi_1 + \pi)$

B_4 $(\exists j \mid 3 \leqslant j \leqslant n)$ $(\varphi_1 \leqslant \varphi_2 \leqslant \ldots < \varphi_{j-1} < \varphi_j = \ldots = \varphi_n = \varphi_1 + \pi)$

It is possible to show that condition $A(\mu)$ is equivalent to the following statement of a geometrical nature.

Lemma 3.3. For condition $A(\mu)$ to be satisfied it is necessary and sufficient that there exist such points $\varphi_{s_1}(\mu)$, $\varphi_{s_2}(\mu)$ and $\varphi_{s_3}(\mu)$ that the triangle formed by these is acute.

Condition (3.3) is the algebraic criterion of acuteness of $\triangle\, \varphi_{s_1}\varphi_{s_2}\varphi_{s_3}$. In case B_1 all nondegenerate triangles are abtuse, and in case B_4 there is at least one right angle among them.

It follows from the above lemmas (without proof) and from the continuity of functions $\varphi_s(\mu)$ that when conditions $A(\mu_0)$ and $B_1(\mu_0)$ are satisfied, then for a fairly small ε conditions $A(D)$ and $B_1(D)$ are also satisfied. Conditions $B_2(\mu_0)$, $B_3(\mu_0)$ and $B_4(\mu_0)$ cannot be maintained when μ is varied. Case B_2 may convert to B_1, while B_3 and B_4 to A or B_1.

As shown in [2, 8] system (3.1) has in the majority of cases when $\mu = \mu_0$, a set of integrals of the form

$$V_0 = \sum_{s=1}^{n} \gamma_s^\circ \omega_s, \quad \omega_s = u_s v_s, \quad \gamma_s^\circ = \text{const} \tag{3.5}$$

The necessary and sufficient condition of stability of system (3.1) when $\mu = \mu_0$ is the presence among (3.5) of integrals of fixed sign. In the case of instability all integrals (3.5) are with alternating signs. In case B_1 when $n = 2$ the system is unstable and has no integrals of the form (3.5). The necessary and sufficient condition of existence among (3.5) of fixed sign integrals is the fulfilment of condition $A(\mu_0)$ or $B_3(\mu_0)$. In the unstable case B_4 there are fixed sign integrals among (3.5).

Let us consider matrix C and the n-dimensional row vectors a and b

$$C = \left\| \begin{matrix} a_1 a_2 \ldots a_n \\ b_1 b_2 \ldots b_n \end{matrix} \right\|, \quad a = (a_1, \ldots, a_n), \quad b = (b_1, \ldots, b_n)$$

The equality $D_{s_j s_k} = |\alpha_{s_j} \alpha_{s_k}| \sin(\varphi_{s_k} - \varphi_{s_j})$ implies that when conditions $A(\mu)$, $B_1(\mu)$ and $B_4(\mu)$ are satisfied, we have $\operatorname{Rank} C(\mu) = 2$, while in cases B_2 and B_3 we have $\operatorname{Rank} C(\mu) = 1$.

Theorem 3.1. If matrix C maintains its rank in region D and the number of nonzero components of vector a or b exceeds the rank of that matrix, system (3.1) has the following set of integrals continuous with respect to μ :

$$V(\mu, \omega) = \sum_{s=1}^{n} \gamma_s(\mu)\,\omega_s, \quad \mu \in D, \quad V(\mu_0, \omega) = V_0 \tag{3.6}$$

The necessary and sufficient condition of the presence among these of integrals that are continuous in D is the fulfilment of condition $A(\mu_0)$ or $B_3(\mu_0)$.

Proof. First of all it is possible to ascertain that the assumption about the rank of the matrix is satisfied when conditions $A(\mu_0)$, $B_1(\mu_0)$ and $B_4(\mu_0)$ are satisfied. It follows from this assumption that when conditions $B_2(\mu_0)$ and $B_3(\mu_0)$ are satisfied, then conditions $B_2(D)$ and $B_3(D)$ are satisfied, since the possible conversion of cases B_2 and B_3 to A or B_1 for $\mu \neq \mu_0$ is associated with a change of the rank of matrix C.

We determine the derivative of function V by (3.6) and by virtue of system (3.1) and from the equation $V^\cdot = 0$ we obtain for the determination of $\gamma_s(\mu)$ the system of equations

$$C(\mu)\,\gamma(\mu) = 0, \quad \gamma(\mu) = \text{colomn}\,[\gamma_1(\mu), \ldots, \gamma_n(\mu)] \tag{3.7}$$

which is compatible with conditions of the theorem.

Let initially $\operatorname{Rank} C(\mu) = 2$. We denote vectors a, b and γ in which there are no components with subscripts s_1, s_2 and s_3 by a', b' and γ'. The general solution of (3.7) can be represented in the form

$$\gamma_{s_1} = D_{s_1s_2}^{-1}(\gamma_{s_3}D_{s_3s_2} + D_{s_2\rho}), \quad \gamma_{s_2} = D_{s_1s_2}^{-1}(\gamma_{s_3}D_{s_2s_1} + D_{s_2\rho}) \tag{3.8}$$

$$\left(D_{s_1s_2} \neq 0, \; D_{s_j\rho} = \begin{vmatrix} a_{s_j} & \rho_1 \\ b_{s_j} & \rho_2 \end{vmatrix}, \quad \rho_1 = -a'\gamma', \quad \rho_2 = -b'\gamma'\right)$$

where s_1 and s_2 are selected so that the conditions appearing in parentheses are satisfied, and the components of vectors γ' and γ_{s_3} are free parameters of the solution.

If conditions $A(\mu_0)$ are satisfied, there exist such s_1, s_2 and s_3 for which formulas (3.3) valid at point μ_0 are preserved also in region D. It follows directly from (3.8) that positive solutions of (3.7) exist only when conditions (3.3) are satisfied.

The free parameters which provide strictly positive solutions of the system (and by the same token fixed sign integrals of set (3.6)) must satisfy conditions

$$\gamma_s > 0 \quad (s \neq s_1, s_2, s_3), \quad \gamma_{s_3} > \max\{D_{s_2s_3}^{-1}D_{s_2\rho},\; D_{s_3s_1}^{-1}D_{s_1\rho},\; 0\} \tag{3.9}$$

When conditions $B_1(\mu_0)$ or $B_4(\mu_0)$ are satisfied, formulas (3.3) do not hold for arbitrary s_1, s_2 and s_3, system (3.7) has no strictly positive solutions, and (3.6) has no fixed sign integrals. (All integrals have alternating signs when condition $B_1(\mu_0)$ is satisfied.)

Let now $\operatorname{Rank} C = 1$. We assume that $a \neq 0$, and instead of system (3.7) consider the equation

$$a\gamma = 0 \tag{3.10}$$

If condition $B_3(\mu_0)$ is satisfied, at least one pair of coefficients α_{s_1} and α_{s_2} satisfy at point μ_0 the equality

$$\operatorname{sign} a_{s_1}a_{s_2} = -1 \tag{3.11}$$

which is valid in D. The general solution of system (3.10) with condition (3.11) is of the form

$$\gamma_s = -a_{s_1}^{-1}\Big(a_{s_2}\gamma_{s_2} + \sum_{s \neq s_1, s_2} \gamma_s a_s\Big) \tag{3.12}$$

Selecting $\gamma_s\ (s \neq s_1)$ in conformity with the condition

$$\gamma_s > 0 \quad (s \neq s_1, s_2), \quad \gamma_{s_1} > \max\left\{-\sum_{s\neq s_1, s_2} \gamma_s a_s, 0\right\}$$

we obtain a strictly positive solution of Eq. (3.10) and a fixed sign integral of system (3.1). When condition $B_2(\mu_0)$ is satisfied, condition (3.11) is violated for any arbitrary s_1 and s_2, and (3.10) has no positive solution. By selecting all parameters $\gamma_s(\mu)$ in the form of continuous functions in D it is possible to assert that the derived integrals are continuous with respect to μ.

Using the set of continuous integrals, we obtain below the criteria of strong stability of system (2.1).

4. Strong stability. Bifurcation. Besides system (2.10), which is equivalent as regards stability to system (2.1), we consider in region D also the system

$$u_s^{*\cdot} = \lambda_s(\mu) u_s^* + u_s^* \sum_{|p|=N} \alpha_s^p(\mu) \omega^{*p} + O_\mu^*(\|u+v\|^{2N+2}) \qquad (4.1)$$

$$v_s^{*\cdot} = -\lambda_s(\mu) v_s^* + v_s^* \sum_{|p|=N} \bar{\alpha}_s^p(\mu) \omega^{*p} + O_\mu^*(\|u+v\|^{2N+2})$$

which is equivalent to system (2.1) in region D^*. System (4.1) is obtained by the usual normalization of system (2.1) in region D^*.

As shown in Sect. 2, coefficients α_s^p in systems (2.10) and (4.1) are the same for $k \geqslant 3$. It is important to note that when in (2.10) the nonlinearities O_μ are continuous in D, then owing to the coefficients at terms of order not lower than $2k-1$ in system (4.1) $O_\mu^* \to \infty$ when $\mu \to \mu_0$. The coefficients α_s^p ($|p| = 1$) have this property even at $k = 2$. Because of this (4.1) is considered here only for $k > 2$.

Let system (3.1) have a continuous set of integrals (3.6). For calculating the derivative of function V we take into account that V is the integral of system (3.1) and, by virtue of system (2.10), we have

$$\frac{1}{2} V^{\cdot} = \sum_{|q|=N+1} T_q(\mu) \omega^q + O_\mu(\|\omega\|^{N+1/2}) = W_{N+1}(\mu, \omega) + O_\mu \qquad (4.2)$$

$$T_q(\mu) = \sum_{s=1}^{n} \gamma_s(\mu) \operatorname{Re} \alpha_s^{q-\delta_s}$$

If the ($N+1$)-st order form $W_{N+1}(\mu_0, \omega) = \overset{\circ}{W}_{N+1}$ is of fixed sign in cone $\omega \geqslant 0$, then for a reasonably small ε function $V^{\cdot}$ is of fixed sign in the neighborhood of zero for all $\mu \in D = U_\varepsilon(\mu_0)$.

We denote by Γ_0 the set of vectors $\gamma(\mu_0)$ for which the form $\overset{\circ}{W}_{N+1}$ is negative definite (obviously Γ_0 can also be $\varnothing$). We further denote by M_0 the set of solutions of system (3.7) when $\mu = \mu_0$, and by M_0^+ the set of strictly positive solutions. Note that when conditions $A(\mu_0)$ or $B_3(\mu_0)$ are satisfied, $M_0^+ \neq \varnothing$.

Theorem 4.1. Let system (2.10) be such that: (1) Rank C is maintained in D for reasonably small ε, and (2) $\Gamma_0 \neq \varnothing$. Then

a) system (2.10) is strongly asymptotically stable at point μ_0, if conditions $A(\mu_0)$ or $B_3(\mu_0)$ are satisfied and $\Gamma_0 \cap M_0^+ \neq \varnothing$;

b) system (2.10) is strongly unstable at point μ_0, if one of conditions $A(\mu_0)$ or $B_3(\mu_0)$ is satisfied and $\Gamma_0 \cap (M_0 \setminus M_0^+) \neq \varnothing$.

The validity of this theorem follows from Theorem 3.1 and the theorems of Liapunov's second method. Let us, for example, consider case (a). Condition (2) implies the existence of vector γ° such that the form $\overset{\circ}{W}_{N+1}$ is negative definite. Condition (a) ensures the existence in set (3.5) of fixed sign integral V_0 of system (3.1) whose derivative by virtue of (2.10) is negative definite (which is determined by the form $\overset{\circ}{W}_{N+1}$). Condition (1) and Theorem 3.1 allow us to assert the existence of Liapunov's function that is continuous with respect to μ and satisfies at every point of region D the conditions of Liapunov's theorem on asymptotic stability.

Case (b) in which the alternating sign integrals of system (3.1) are used, is similarly considered.

Let us now deal with the question of properties of system (2.10) in certain cases when the conditions of Theorem 4.1 are violated. For this we shall consider system (4.1) in D^* and the function

$$V^* = \sum_{s=1}^{n} \gamma_s(\mu)\,\omega_s^*$$

where γ_s are arbitrary continuous functions of μ (V^*, unlike V in (3.6), is not necessarily an integral of system (3.1)).

Taking into account that α_s^p are the same in systems (2.10) and (4.1), we have

$$V^{*\cdot} = W_{N+1}(\mu, \omega^*) + O_\mu^*(\|\omega\|^{N+{}^3/_2}) \tag{4.3}$$

where $W_{N+1}(\mu, \omega^*)$ is a form analogous to (4.2) and $O_\mu^* \to \infty$ when $\mu \to \mu_0$.

Let the set Γ_0 for form $W_{N+1}(\mu, \omega^*)$ be nonempty. If Γ_0 contains the strictly positive vector γ°, system (4.1) is asymptotically stable in D^*. Independently of this property of system (4.1), any of the cases A and B_j can be realized at point μ_0 for system (3.1). If it is assumed that conditions $B_1(\mu_0)$ or $B_2(\mu_0)$ are satisfied, system (2.10) is then unstable when $\mu = \mu_0$ [2]. It is obvious that on these assumptions $\Gamma_0 \cap M_0 = \varnothing$, and none of conditions (a) and (b) are satisfied.

The above considerations lead to the following result.

Theorem 4.2. If system (2.10) is such that conditions $B_1(\mu_0)$ or $B_2(\mu_0)$ are satisfied, and form $W_{N+1}(\mu_0, \omega^*)$ can be made negative definite, then μ_0 is a bifurcation point at which the asymptotic stability in D^* changes to instability. If the set $\Gamma_0 \neq \varnothing$ does not contain strictly positive vectors, then system (2.10) is strongly unstable in D when conditions $B_1(\mu_0)$ or $B_2(\mu_0)$ are satisfied.

Note that the region in which function $V^{*\cdot}$ is of fixed sign contracts under conditions of Theorem 4.2 when $\mu \to \mu_0$ to the coordinate origin, as implied by the properties of function O_μ^* in (4.3). When $\mu = \mu_0$ an instability region, lying in the neighborhood of the unstable solution of system (3.1), is generated [2].

Example. Let us consider the system of differential equations

$$z_s^{\cdot\cdot} + \rho_s^2(\mu) z_s = Z_s^{(2N)}(\mu, z, z^{\cdot 2}) + Z_s^{2N+1}(\mu, z, z^{\cdot 2}) + g_s(\mu)\, z_s^{\cdot 2N+1} + \ldots \tag{4.4}$$
$$(s = 1, 2, \ldots, n,\ N > 1)$$

where $\rho_s(\mu)$, $g_s(\mu)$ and the coefficients of forms $Z_s^{(j)}$ are continuous functions of μ.

Using the substitution $x_s = z_s - (i/\rho_s) z_s^{\cdot} = y_s$ followed by the transformation to the continuous normal form, we reduce the problem to the analysis of system (2.10) whose coefficients are defined as follows:

$$\lambda_s = i\rho_s, \qquad \alpha_s(\mu) = -\frac{i}{\rho_s(\mu)}\, b_s(\mu), \qquad \alpha_s^p(\mu) = -\frac{i}{\rho_s(\mu)}\, b_s^p(\mu) \text{ for } p \neq N\delta_s$$

$$\alpha_s^{N\delta_s}(\mu) = \rho_s^{2N} 2^{-2N-1} C_{2N+1}^N g_s - \frac{i}{\rho_s} b_s^{N\delta_s}$$

The real numbers $b_s(\mu)$ and $b_s^p(\mu)$ are coefficients at terms of inner and identical resonances in forms $Z^{(j)}(\mu, {}^1/_2(x_s + y_s), -{}^1/_4\rho_s^2(y_s - x_s)^2)$ for $j = 2N$ and $2N + 1$, respectively. At point μ_0 the numbers $b_s(\mu_0) = b_s^\circ$ and $g_s(\mu_0) = g_s^\circ$ are nonzero.

It is seen that for system (4.4) matrix $C(\mu)$ retains its rank in D, and that at point μ_0 either conditions $B_2(\mu_0)$ or $B_3(\mu_0)$ are satisfied when either all b_s°/ρ_s° are of the same sign or when among b_s°/ρ_s° there are numbers of different signs, respectively.

Form $\overset{\circ}{W}_{N+1}$ and the sets Γ_0 and M_0 are defined as follows:

$$\overset{\circ}{W}_{N+1} = \sum_{s=1}^{n} \gamma_s^\circ \operatorname{Re} \alpha_s^{N\delta_s}(\mu_0)\,\omega_s^{N+1} \qquad (\operatorname{sign} \operatorname{Re} \alpha^{N\delta_s}(\mu_0) = \operatorname{sign} g_s^\circ)$$

$$\Gamma_0 = \{\gamma^\circ \mid \operatorname{sign} \gamma_s^\circ = -\operatorname{sign} g_s^\circ\}, \quad M_0 = \left\{\gamma^\circ \mid \sum_{j=1}^{n} b_j^\circ \gamma_j^\circ = 0\right\}$$

It is evident that $\Gamma_0 \neq \varnothing$ and conditions (1) and (2) of Theorem 4.1 are satisfied for system (4.4).

When all $g_s^\circ < 0$ and condition $B_3(\mu_0)$ is satisfied, system (4.4) is strongly asymptotically stable (conditions (a) of the theorem are satisfied).

If among g_s° there are numbers of different signs or all $g_s^\circ > 0$, then for certain k, j will be $\operatorname{sign} g_k^\circ b_k^\circ = -\operatorname{sign} g_j^\circ b_j^\circ$, condition (b) of Theorem 4.1 is satisfied, and system (4.4) is strongly unstable. The last requirement is automatically realized when all b_s°/ρ_s° are of the same sign and among g_s° there are numbers of different signs, or vice versa.

If all $g_s^\circ < 0$ and condition $B_2(\mu_0)$ is satisfied, then by Theorem 4.2 at point μ_0 the bifurcation — the asymptotic stability in D^* — changes to instability at that point.

Finally, if condition $B_2(\mu_0)$ is satisfied and $(\forall s) g_s^\circ b_s^\circ/\rho_s^\circ < 0\ (> 0)$, by Theorem 4.2 we have strong instability at point μ_0.

We note in conclusion that Theorem 4.2 may be amplified as follows.

T h e o r e m 4.3. If conditions $B_1(\mu_0)$ or $B_2(\mu_0)$ are satisfied for system (2.10) and system (4.1) is asymptotically stable in D^* within terms of order not higher than $2k - 2$, then μ_0 is the bifurcation point of (2.10) of the same kind as in Theorem 4.2.

If system (4.1) is unstable in D^* to within terms of order not higher than $2k-2$ and conditions $B_1(\mu_0)$ or $B_2(\mu_0)$ are satisfied, then system (2.10) is strongly unstable at point μ_0.

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